数理解析研究所講究録
第 1874 巻 2014 年 98-111



# Poly-Cauchy numbers and poly-Bernoulli numbers

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## 1 Introduction

The *Cauchy numbers* of the first kind, denoted by $c_n$ ([5]), are defined by the integral of the falling factorial:

$$c_n = \int_0^1 x(x-1)\dots(x-n+1)dx\,.$$

The generating function of the Cauchy numbers of the first kind $c_n$ is given by

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^\infty c_n \frac{x^n}{n!}$$

([23]).

Cauchy numbers are not so famous, though they seem to have similar properties to those of the *Bernoulli numbers.* The classical Bernoulli numbers $B_n$ are defined by the generating function

$$\frac{x}{e^x-1} = \sum_{n=0}^\infty B_n \frac{x^n}{n!} \quad \left(B_1 = -\frac{1}{2}\right)\,.$$

Before the terminology of Cauchy numbers appeared in Comtet's book ([5]), the concept of the Cauchy numbers was first introduced by Nörlund ([24, pp.146–147]) in 1924. Here, the higher order Bernoulli numbers $B_n^{(r)}$ are defined by

$$\left(\frac{x}{e^x-1}\right)^r = \sum_{n=0}^\infty B_n^{(r)} \frac{x^n}{n!} \quad (|x| < 2\pi)$$

[1]This research was supported in part by the Grant-in-Aid for Scientific research (C) (No.22540005), the Japan Society for the Promotion of Science.

or

$$\left(\frac{\ln(1+x)}{x}\right)^r = r\sum_{n=0}^{\infty}\frac{B_n^{(r+n)}}{r+n}\frac{x^n}{n!} \quad (|x|<1)\,.$$

See also [8, p.257,p.259]. Then

$$B_n^{(n)} = \int_0^1 (x-1)(x-2)\cdots(x-n)dx$$

or

$$B_{n+1}^{(n)} = -n\int_0^1 x(x-1)\cdots(x-n)dx\,.$$

Hence, $c_n = -B_n^{(n-1)}/(n-1)$. Ch. Jordan studied the Bernoulli numbers of the second kind $b_n$ ([13, p.131]), defined by

$$b_n = \psi_{n+1}(1) - \psi_{n+1}(0) = \int_0^1 \binom{x}{n}dx\,.$$

Hence, $b_n = c_n/n!$. In 1961 Carlitz ([4]) introduced the numbers $\beta_n$, defined by

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^{\infty}\beta_n\frac{x^n}{n!}$$

Namely, $\beta_n = c_n$.

Cauchy numbers and Bernoulli numbers are much related to the Stirling numbers of the first kind and of the second kind. The (unsigned) Stirling numbers of the first kind $\left[{n \atop m}\right]$ arise as coefficients of the rising factorial

$$x(x+1)\dots(x+n-1) = \sum_{m=0}^{n}\left[{n \atop m}\right]x^m\,.$$

The Stirling numbers of the second kind $\left\{{n \atop m}\right\}$ are determined by

$$\left\{{n \atop m}\right\} = \frac{1}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}(m-j)^n\,.$$

There are many identities about the Bernoulli numbers. They are much related to the (unsigned) Stirling numbers of the first kind $\left[{n \atop m}\right]$ and the

Stirling numbers of the second kind $\left\{ n \atop m \right\}$. Some of them are

$$\frac{1}{n!}\sum_{m=0}^{n}(-1)^m \begin{bmatrix} n+1 \\ m+1 \end{bmatrix} B_m = \frac{1}{n+1},$$

$$B_n = (-1)^n \sum_{m=0}^{n} \left\{ n \atop m \right\} \frac{(-1)^m m!}{m+1}.$$

The corresponding identities of the classical Cauchy numbers are

$$\sum_{m=0}^{n} \left\{ n \atop m \right\} c_m = \frac{1}{n+1},$$

$$c_n = (-1)^n \sum_{m=0}^{n} \left[ n \atop m \right] \frac{(-1)^m}{m+1}.$$

# 2 Polylogarithms

The $k$-th *polylogarithm* function is defined by

$$\mathrm{Li}_k(x) = \sum_{m=1}^{\infty} \frac{x^m}{m^k}.$$

The $k$-th *polylogarithm factorial* function is defined by

$$\mathrm{Lif}_k(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!(m+1)^k}.$$

For $k \geq 2$

$$x\frac{d}{dx}\mathrm{Li}_k(x) = \mathrm{Li}_{k-1}(x),$$

so

$$\mathrm{Li}_k(x) = \int_0^x \frac{\mathrm{Li}_{k-1}(t)}{t} dt;$$

on the other hand,

$$\frac{d}{dx}(x\mathrm{Lif}_k(x)) = \mathrm{Lif}_{k-1}(x),$$

so

$$\mathrm{Lif}_k(x) = \frac{1}{x}\int_0^x \mathrm{Lif}_{k-1}(t)dt.$$

In special, for $k=0,1$ we have

$$\mathrm{Li}_0(x)=\frac{x}{1-x}, \quad \mathrm{Li}_1(x)=-\ln(1-x)$$

and

$$\mathrm{Lif}_0(x)=e^x, \quad \mathrm{Lif}_1(x)=(e^x-1)/x\,.$$

For $k=-r$ we have

$$\mathrm{Li}_{-r}(x)=\frac{1}{(1-x)^{r+1}}\sum_{j=0}^{r}\left\langle {r \atop j} \right\rangle x^{r-j} \quad (r=0,1,2,\dots)$$

([3]), where

$$\left\langle {r \atop j} \right\rangle=\sum_{l=0}^{j+1}(-1)^l\binom{r+1}{l}(j-l+1)^r$$

are the Eulerian numbers.

On the other hand, for $k=-r$ we have

$$\mathrm{Lif}_{-r}(x)=e^x\sum_{j=0}^{r}\left\{ {r+1 \atop j+1} \right\} x^j \quad (r=0,1,2,\dots)\,.$$

We have the record for the first some values $r$.

$$\begin{aligned}
\mathrm{Lif}_0(x)&=e^x,\\
\mathrm{Lif}_{-1}(x)&=(1+x)e^x,\\
\mathrm{Lif}_{-2}(x)&=(1+3x+x^2)e^x,\\
\mathrm{Lif}_{-3}(x)&=(1+7x+6x^2+x^3)e^x,\\
\mathrm{Lif}_{-4}(x)&=(1+15x+25x^2+10x^3+x^4)e^x,\\
\mathrm{Lif}_{-5}(x)&=(1+31x+90x^2+65x^3+15x^4+x^5)e^x.
\end{aligned}$$

In 1997 M. Kaneko ([18]) introduced the *poly-Bernoulli numbers* $B_n^{(k)}$ by

$$\frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}}=\sum_{n=0}^{\infty}B_n^{(k)}\frac{x^n}{n!}\,.$$

When $k=1$, $B_n^{(1)}$ is the classical Bernoulli number with $B_1^{(1)}=1/2$.

Recently, we [19] introduced the *poly-Cauchy numbers* $c_n^{(k)}$ by

$$\mathrm{Lif}_k(\ln(1+x)) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!}\,.$$

When $k=1$, $c_n^{(1)} = c_n$ is the classical Cauchy number.

*Poly-Cauchy numbers of the first kind* $c_n^{(k)}$ may be defined by

$$c_n^{(k)} = \underbrace{\int_0^1 \dots \int_0^1}_{k} (x_1x_2\dots x_k)(x_1x_2\dots x_k - 1)$$
$$\cdots (x_1x_2\dots x_k - n + 1)dx_1dx_2\dots dx_k\,.$$

In addition, *poly-Cauchy numbers of the second kind* $\hat{c}_n^{(k)}$ are defined by

$$\hat{c}_n^{(k)} = \underbrace{\int_0^1 \dots \int_0^1}_{k} (-x_1x_2\dots x_k)(-x_1x_2\dots x_k - 1)$$
$$\cdots (-x_1x_2\dots x_k - n + 1)dx_1dx_2\dots dx_k\,.$$

The generating function of the poly-Bernoulli numbers are written in terms of iterated integrals:

$$\frac{e^x}{e^x-1} \underbrace{\int_0^x \frac{1}{e^x-1} \cdots \int_0^x \frac{1}{e^x-1}}_{k-1} \times x \underbrace{dx\dots dx}_{k-1} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!}\,.$$

An explicit formula for $B_n^{(k)}$ is given by

$$B_n^{(k)} = (-1)^n \sum_{m=0}^{n} \left\{ n \atop m \right\} \frac{(-1)^m m!}{(m+1)^k} \quad (n \ge 0, k \ge 1)\,. \tag{1}$$

The generating function of the poly-Cauchy numbers can be also written in the form of iterated integrals:

$$\frac{1}{\ln(1+x)} \underbrace{\int_0^x \frac{1}{(1+x)\ln(1+x)} \cdots \int_0^x \frac{1}{(1+x)\ln(1+x)}}_{k-1} \times x \underbrace{dx\dots dx}_{k-1}$$
$$= \sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!}\,.$$

An explicit formula for $c_n^{(k)}$ is given by

$$c_n^{(k)} = (-1)^n \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-1)^m}{(m+1)^k}\,. \tag{2}$$

There are some relations between poly-Cauchy numbers and poly-Bernoulli numbers.

**Theorem 1.** *For $n \geq 1$ we have*

$$B_n^{(k)} = \sum_{l=1}^{n}\sum_{m=1}^{n} m! \left\{ {n \atop m} \right\} \left\{ {m-1 \atop l-1} \right\} c_l^{(k)}\,,$$

$$c_n^{(k)} = \sum_{l=1}^{n}\sum_{m=1}^{n} \frac{(-1)^{n-m}}{m!} \begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ l \end{bmatrix} B_l^{(k)}\,.$$

# 3 Duality theorem

It is known that the duality theorem holds for poly-Bernoulli numbers ([18]). Namely,

$$B_n^{(-k)} = B_k^{(-n)} \quad (n,\ k \geq 0)\,.$$

It is due to the symmetric formula:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B_n^{(-k)} \frac{x^n}{n!}\frac{y^k}{k!} = \frac{e^{x+y}}{e^x + e^y - e^{x+y}}\,.$$

It follows that

$$B_n^{(-k)} = \sum_{m=0}^{n} (-1)^{m+n} m! \left\{ {n \atop m} \right\} (m+1)^k\,,$$

$$B_n^{(-k)} = \sum_{j=0}^{k} (j!)^2 \left\{ {n+1 \atop j+1} \right\} \left\{ {k+1 \atop j+1} \right\}.$$

However, the duality theorem does not hold for poly-Cauchy numbers. In fact, we have

**Proposition 1.**

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} c_n^{(-k)} \frac{x^n}{n!}\frac{y^k}{k!} = e^y(1+x)^{e^y}\,,$$
$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \hat{c}_n^{(-k)} \frac{x^n}{n!}\frac{y^k}{k!} = \frac{e^y}{(1+x)^{e^y}}\,.$$

By using Proposition 1 we have explicit expressions of the poly-Cauchy numbers with negative indices.

**Theorem 2** ([16]).

$$c_n^{(-k)} = \sum_{j=0}^{k}(-1)^{n+j}j!\left(\begin{bmatrix}n\\j\end{bmatrix} - n\begin{bmatrix}n-1\\j\end{bmatrix}\right)\begin{Bmatrix}k+1\\j+1\end{Bmatrix}\,,$$
$$\hat{c}_n^{(-k)} = \sum_{j=0}^{k}(-1)^{n}j!\begin{bmatrix}n+1\\j+1\end{bmatrix}\begin{Bmatrix}k+1\\j+1\end{Bmatrix}\,.$$

Moreover, using Theorem 2 with (2) we have the following congruence results.

**Theorem 3.** *For any positive integer $k$, $c_n^{(-k)} \equiv c_n^{(-k-4)} \pmod{10}$ and $\hat{c}_n^{(-k)} \equiv \hat{c}_n^{(-k-4)} \pmod{10}$. In special, when $n = 1$, for $k \ge 1$, $c_1^{(-k-4)} \equiv c_1^{(-k)} \pmod{30}$ and $\hat{c}_1^{(-k-4)} \equiv \hat{c}_1^{(-k)} \pmod{30}$.*

**Theorem 4.** *For $k \ge 1$ we have*

$$c_n^{(-k)} \equiv \begin{cases} 0 \pmod 2 & \text{if } n = 1 \text{ or } n \ge 4;\\ 1 \pmod 2 & \text{if } n = 2, 3\,,\end{cases}$$
$$\hat{c}_n^{(-k)} \equiv \begin{cases} 0 \pmod 2 & \text{if } n = 1 \text{ or } n \ge 4;\\ 1 \pmod 2 & \text{if } n = 2, 3\,.\end{cases}$$

# 4 Sums of products

Sums of products of Bernoulli numbers

$$\sum_{\substack{i_1+\cdots+i_m=n\\ i_1,\dots,i_m\ge 0}} \frac{n!}{i_1!\cdots i_m!} B_{i_1}\cdots B_{i_m} \quad (m\ge 1,\ n\ge 0)$$

have been considered by many authors (see, e.g. [1, 2, 6]). When $m=2$, one has the famous Euler's identity:

$$\sum_{i=0}^{n}\binom{n}{i} B_i B_{n-i} = -nB_{n-1} - (n-1)B_n \quad (n\ge 1). \tag{3}$$

Kamano ([14]) considered the sums of products of Bernoulli numbers, including poly-Bernoulli numbers

$$S_m^{(k)}(n) := \sum_{\substack{i_1+\cdots+i_m=n\\ i_1,\dots,i_m\ge 0}} \frac{n!}{i_1!\cdots i_m!} B_{i_1}\cdots B_{i_{m-1}} B_{i_m}^{(k)} \quad (m\ge 1,\ n\ge 0).$$

Then, $S_m^{(k)}(n)$ satisfies the following relation:

**Proposition 2.**

$$\sum_{l=0}^{m}(-1)^{m-l}\begin{bmatrix} m+1 \\ l+1 \end{bmatrix} S_{m+1}^{(k-l)}(n) = \begin{cases} \dfrac{n!}{(n-m)!}\displaystyle\sum_{r=0}^{m}\begin{bmatrix} m \\ r \end{bmatrix} B_{n-m+r}^{(k)} & (n\ge m), \\ 0 & (0\le n\le m-1). \end{cases}$$

Kamano also showed the explicit formulae $S_m^{(k)}(n)$ for $m=2,3$. For example, when $m=2$ we have

**Proposition 3.** *For $k\ge 1$ and $n\ge 0$,*

$$S_2^{(0)}(n) = B_n^{(1)},$$

$$S_2^{(k)}(n) = B_n^{(1)} - n\sum_{j=1}^{k} B_n^{(j)},$$

$$S_2^{(-k)}(n) = B_n^{(1)} + n\sum_{j=0}^{k-1} B_n^{(-j)}.$$

It seemed to be difficult to give an explicit formula for $S_m^{(k)}(n)$ for $m \ge 4$, but recently a general formula for all $m \ge 1$ is given.

**Theorem 5** ([20]). *For $m \ge 1$, $n \ge 0$ and $k \ge 1$, we have*

$$S_{m+1}^{(0)}(n) = S_m^{(1)}(n),$$

$$\begin{aligned} S_{m+1}^{(k)}(n) &= \sum_{r=0}^{m-1}(-1)^r r!\binom{n}{r}\sum_{i=0}^{r}\binom{r}{i}\frac{(-1)^i}{(i+1)^k}S_{m+1-r}^{(0)}(n-r)\\ &\quad + (-1)^m\binom{n}{m}\sum_{\substack{j_1+\cdots+j_m\le k-1\\ j_1,\dots,j_m\ge 0}}\frac{1}{2^{j_2}\cdots m^{j_m}}\sum_{\nu=1}^{m}\begin{bmatrix} m\\ \nu\end{bmatrix}B_{n-m+\nu}^{(1+j_1)},\end{aligned}$$

$$\begin{aligned} &S_{m+1}^{(-k)}(n)\\ &= \sum_{r=0}^{m-1}(-1)^r r!\binom{n}{r}\sum_{i=0}^{r}\binom{r}{i}(-1)^i(i+1)^k S_{m+1-r}^{(0)}(n-r)\\ &\quad + \binom{n}{m}\sum_{\substack{j_1+\cdots+j_m\le k\\ j_1,\dots,j_m\ge 1}}2^{j_2}\cdots m^{j_m}\sum_{\nu=1}^{m}\begin{bmatrix} m\\ \nu\end{bmatrix}B_{n-m+\nu}^{(1-j_1)}.\end{aligned}$$

Sums of products of Cauchy numbers

$$\sum_{\substack{i_1+\cdots+i_m=n\\ i_1,\dots,i_m\ge 0}}\frac{n!}{i_1!\cdots i_m!}c_{i_1}\cdots c_{i_m}\quad (m\ge 1,\ n\ge 0)$$

were studied by Zhao ([25]). Consider the sums of products of Cauchy numbers, including poly-Cauchy numbers

$$T_m^{(k)}(n) := \sum_{\substack{i_1+\cdots+i_m=n\\ i_1,\dots,i_m\ge 0}}\frac{n!}{i_1!\cdots i_m!}c_{i_1}\cdots c_{i_{m-1}}c_{i_m}^{(k)}\quad (m\ge 1,\ n\ge 0)\,.$$

Then, $T_m^{(k)}(n)$ satisfies the following relation:

**Proposition 4** ([22]).

$$\begin{aligned} &\sum_{l=0}^{m}(-1)^{m-l}\begin{bmatrix} m+1\\ l+1\end{bmatrix}T_{m+1}^{(k-l)}(n)\\ &\qquad = \begin{cases} \displaystyle\sum_{l=0}^{m}\sum_{i=0}^{n-m}\frac{n!}{i!}\binom{l}{n-m-i}\left\{\begin{matrix} m\\ l\end{matrix}\right\}c_{l+i}^{(k)} & (n\ge m);\\ 0 & (0\le n\le m-1)\,.\end{cases}\end{aligned}$$

When $m = 2$, we have the following explicit formulae.

**Proposition 5** ([22]). *For $n \ge 0$ and $k \ge 1$ we have*

$$\begin{aligned} T_2^{(0)}(n) &= c_n + nc_{n-1}\,, \\ T_2^{(k)}(n) &= T_2^{(0)}(n) - n\sum_{j=1}^{k}(c_n^{(j)} + (n-1)c_{n-1}^{(j)})\,, \\ T_2^{(-k)}(n) &= T_2^{(0)}(n) + n\sum_{j=0}^{k-1}(c_n^{(-j)} + (n-1)c_{n-1}^{(-j)})\,. \end{aligned}$$

Putting $k = 1$ in the second identity, we have

**Corollary 1** ([25]).

$$\sum_{i=0}^{n}\binom{n}{i}c_i c_{n-i} = -n(n-2)c_{n-1} - (n-1)c_n \quad (n \ge 0)\,.$$

This is an analogue of Euler's identity (3).
In general, we can obtain the following explicit expression of $T_m^{(k)}(n)$ for any general $m \ge 2$.

**Theorem 6.** *For $n \ge 0$ and $k > 0$ we have*

$$\begin{aligned} T_m^{(0)}(n) &= T_{m-1}^{(1)}(n) + nT_{m-1}^{(1)}(n-1)\,, \\ T_m^{(k)}(n) &= \sum_{r=0}^{m-2}(-1)^r\binom{n}{r}\sum_{i=0}^{r}\binom{r}{i}\frac{(-1)^i}{(i+1)^k}T_{m-r}^{(0)}(n-r) \\ &+ \frac{(-1)^{m-1}n!}{(n-m+1)!}\sum_{\substack{j_1+j_2+\cdots+j_{m-1}=k+m-2\\ j_1,j_2,\dots,j_{m-1}\ge 1}} 2^{-j_2}3^{-j_3}\dots(m-1)^{-j_{m-1}}\sum_{j=1}^{j_1}\sum_{\kappa=0}^{m-1}P_{m,\kappa}(n)c_{n-\kappa}^{(j)}\,, \end{aligned}$$

$$\begin{aligned} T_m^{(-k)}(n) &= \sum_{r=0}^{m-2}(-1)^r\binom{n}{r}\sum_{i=0}^{r}\binom{r}{i}(-1)^i(i+1)^k T_{m-r}^{(0)}(n-r) \\ &+ \frac{n!}{(n-m+1)!}\sum_{\substack{j_1+j_2+\cdots+j_{m-1}=k-m+1\\ j_1,j_2,\dots,j_{m-1}\ge 0}} 2^{j_2}3^{j_3}\dots(m-1)^{j_{m-1}}\sum_{j=0}^{j_1}\sum_{\kappa=0}^{m-1}P_{m,\kappa}(n)c_{n-\kappa}^{(-j)}\,. \end{aligned}$$

*where*

$$P_{m,\kappa}(n)=\sum_{t=0}^{\kappa}\left\{ {m-1 \atop m-t-1} \right\}\binom{m-t-1}{m-\kappa-1}\frac{(n-m+1)!}{(n-m-\kappa+t+1)!}$$
$$(\kappa=0,1,\dots,m-2)$$

*and*

$$\begin{aligned}P_{m,m-1}(n)&=\sum_{t=0}^{m-2}\left\{ {m-1 \atop m-t-1} \right\}\frac{(n-m+1)!}{(n-2m+t+2)!}\\&=(n-m+1)^{m-1}.\end{aligned}$$

# 5 Hypergeometric Bernoulli numbers and hypergeometric Cauchy numbers

*Hypergeometric Bernoulli numbers* $B_{N,n}$ $(N\ge 1,\ n\ge 0)$ ([7, 9, 10, 11, 12]) are defined by

$$\frac{1}{{}_1F_1(1;N+1;x)}=\frac{x^N/N!}{e^x-\sum_{n=0}^{N-1}x^n/n!}=\sum_{n=0}^{\infty}B_{N,n}\frac{x^n}{n!},$$

where ${}_1F_1(a;b;z)$ is the confluent hypergeometric function defined by

$${}_1F_1(a;b;z)=\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{z^n}{n!}$$

with the Pochhammer symbol $(x)_n=x(x+1)\dots(x+n-1)$ $(n\ge 1)$ and $(x)_0=1$. When $N=1$, $B_{1,n}=B_n$ are classical Bernoulli numbers.

*Hypergeometric Cauchy numbers* $c_{N,n}$ $(N\ge 1,\ n\ge 0)$ ([21]) are defined by

$$\frac{1}{{}_2F_1(1,N;N+1;-x)}=\frac{(-1)^{N-1}x^N/N}{\ln(1+x)-\sum_{n=1}^{N-1}(-1)^{n-1}x^n/n}=\sum_{n=0}^{\infty}c_{N,n}\frac{x^n}{n!},$$

where ${}_2F_1(a,b;c;z)$ is the hypergeometric function defined by

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}.$$

When $N = 1$, $c_{1,n} = c_n$ are classical Cauchy numbers.

We record of the first few values of $c_{N,n}$:

$$
\begin{aligned}
c_{N,0} &= 1,\\
c_{N,1} &= \frac{N}{N+1},\\
c_{N,2} &= -\frac{2N}{(N+1)^2(N+2)},\\
c_{N,3} &= \frac{6N(N^2+N+2)}{(N+1)^3(N+2)(N+3)}\\
c_{N,4} &= -\frac{4!N(N^5+5N^4+14N^3+24N^2+20N+12)}{(N+1)^4(N+2)^2(N+3)(N+4)},\\
c_{N,5} &= \frac{5!N(N^7+8N^6+35N^5+96N^4+160N^3+184N^2+116N+48)}{(N+1)^5(N+2)^2(N+3)(N+4)(N+5)}.
\end{aligned}
$$

The sums of products of hypergeometric Bernoulli numbers were studied by Kamano ([15]) and those of hypergeometric Cauchy numbers are also studied in [21].

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